Fuzzy Sets and Fuzzy Logic

Crisp sets

- Collection of definite, well-definable objects (elements).

Representation of sets:

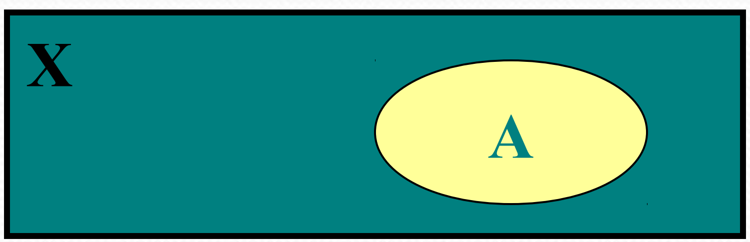
- list of all elements

  $A=\{x_1, \ldots, x_n\}, x_j \in X$

- elements with property P

  $A=\{x | x \text{ satisfies } P\}, x \in X$

- Venn diagram



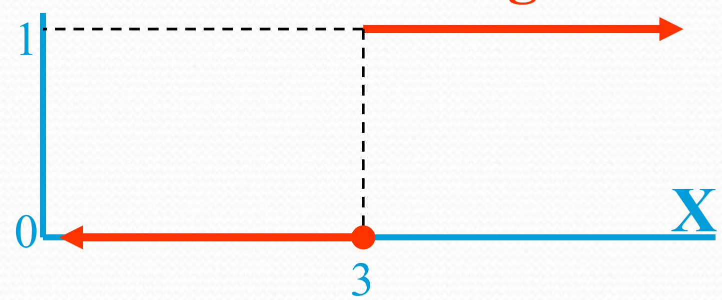
- characteristic function

  $f_A: X \rightarrow \{0,1\},$

  $f_A(x) = 1, \Leftrightarrow x \in A$

  $f_A(x) = 0, \Leftrightarrow x \notin A$

Real numbers larger than 3:



Crisp (traditional) logic

- Crisp sets are used to define interpretations of first order logic

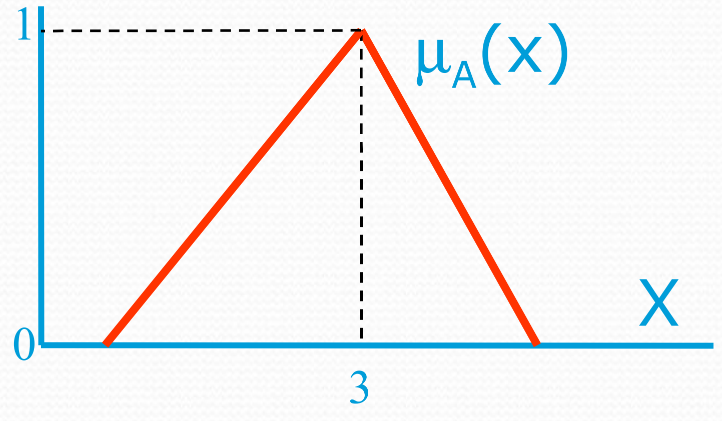
  If P is a unary predicate, and we have no functions, a possible interpretation is
  $A = \{0,1,2\}$
  $P^I = \{0,2\}$
  within this interpretation, $P(0)$ and $P(2)$ are true, and $P(1)$ is false.

- Crisp logic can be "fragile": changing the interpretation a little can change the truth value of a formula dramatically.

Fuzzy sets

- Sets with fuzzy, gradual boundaries (Zadeh 1965)
- A fuzzy set A in X is characterized by its membership function $\mu_A: X \rightarrow [0,1]$

Real numbers about 3:



A fuzzy set A is completely determined by the set of ordered pairs

$A=\{(x,\mu_A(x)) | x \in X\}$

X is called the *domain* or *universe of discourse*

Fuzzy sets on discrete universes

- Fuzzy set C = "desirable city to live in"

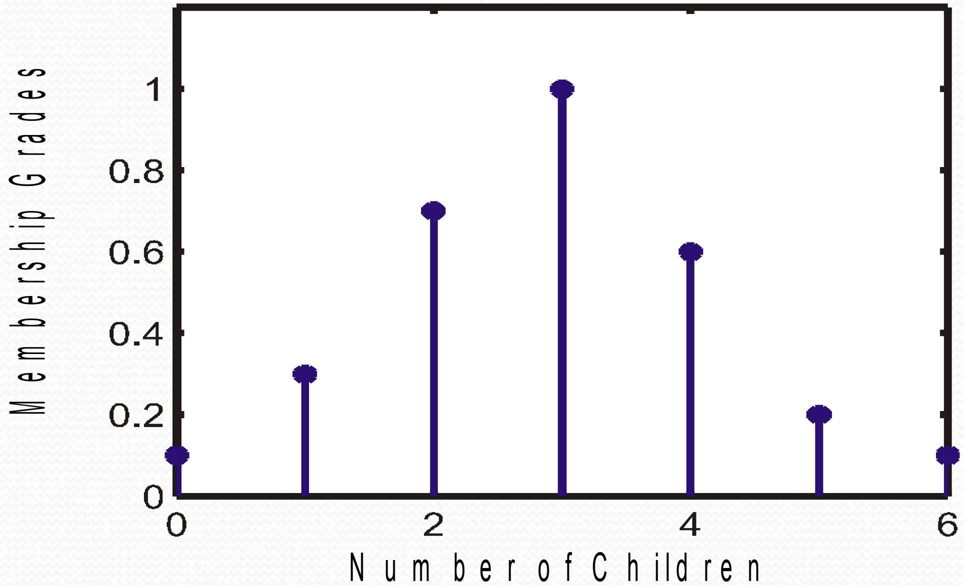
  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = "sensible number of children"

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



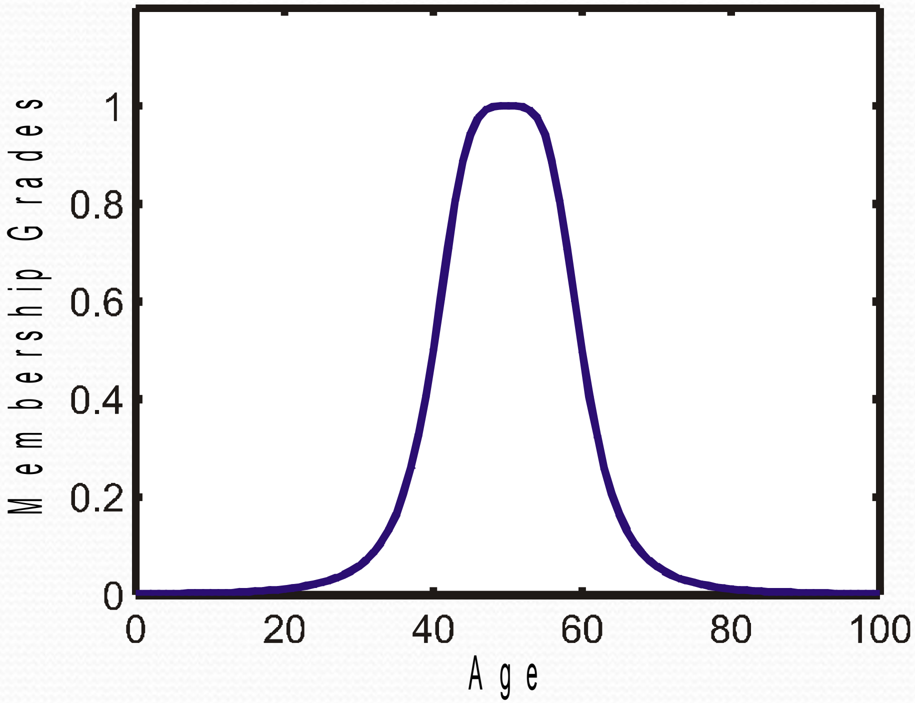
Fuzzy sets on continuous universes

- Fuzzy set B = "about 50 years old"

  X = Set of positive real numbers (continuous)

  B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Membership Function formulation

Triangular MF:

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

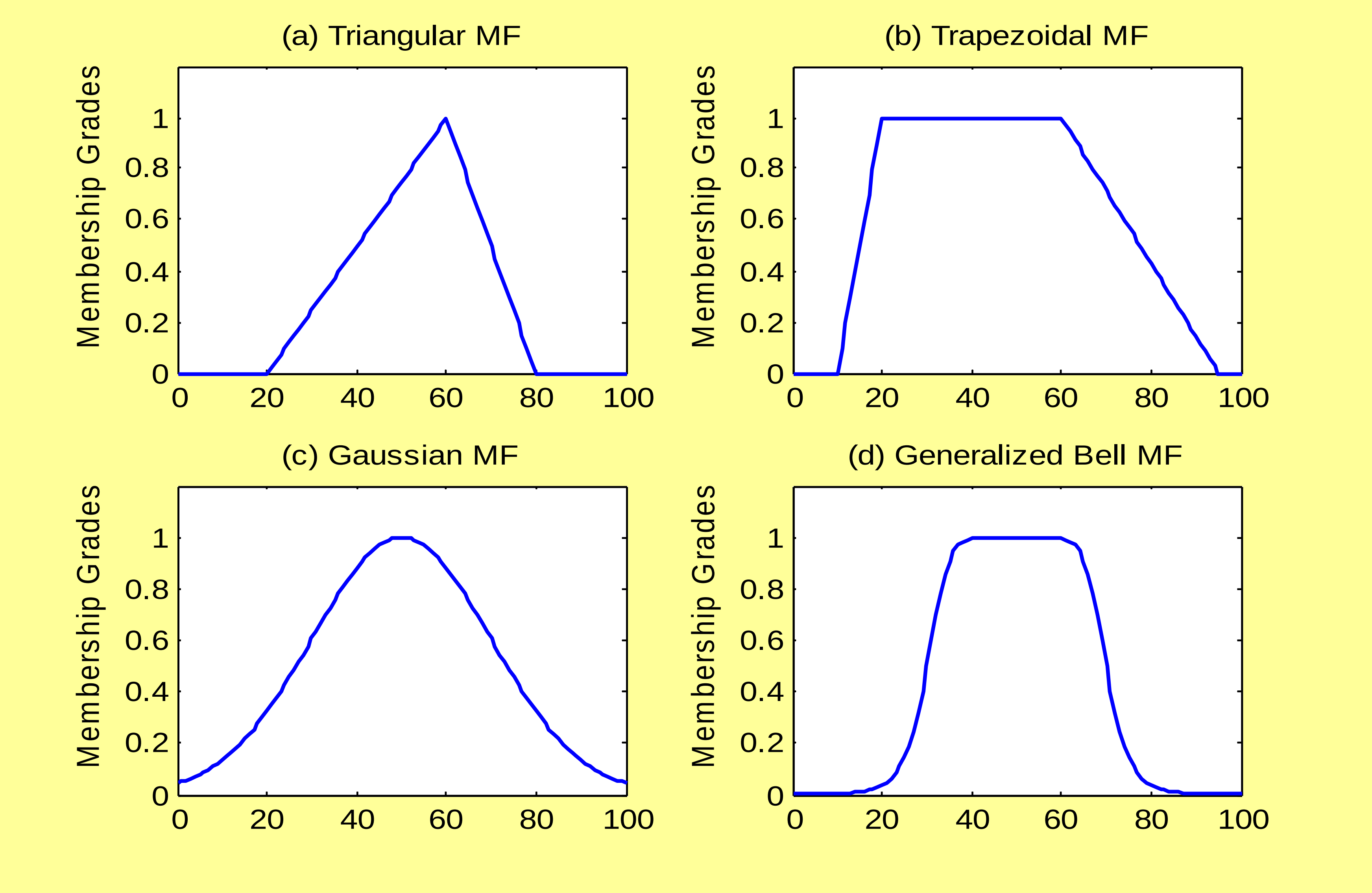
Gaussian MF:

$$gaussmf(x;a,b) = e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$

Generalized bell MF:

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$$

MF formulation



Fuzzy sets & fuzzy logic

- Fuzzy sets can be used to define a level of truth of facts
- Fuzzy set C = "desirable city to live in"

  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

  corresponds to a fuzzy interpretation in which
  C(SF) is true with degree 0.9
  C(Boston) is true with degree 0.8
  C(LA) is true with degree 0.6
  → membership function $\mu_C(x)$ can be seen as a (fuzzy) predicate.

Notation

Many texts (especially older ones) do not use a consistent and clear notation

X is discrete

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

$$A = \sum_{x_i \in X} \mu_A(x_i)\; x_i$$

X is continuous

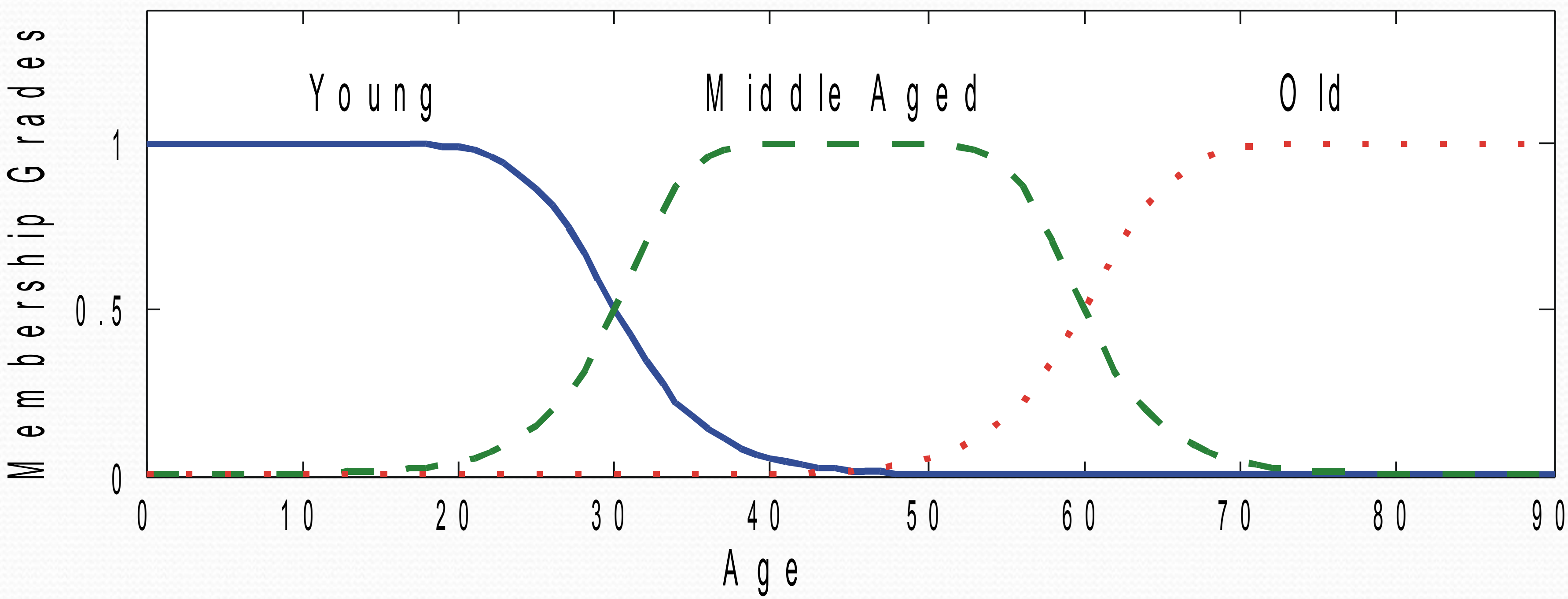
$$A = \int_X \mu_A(x) / x$$

$$A = \int_X \mu_A(x)\; x$$

Note that Σ and integral signs stand for the union of membership grades; “/” stands for a marker and does not imply division.

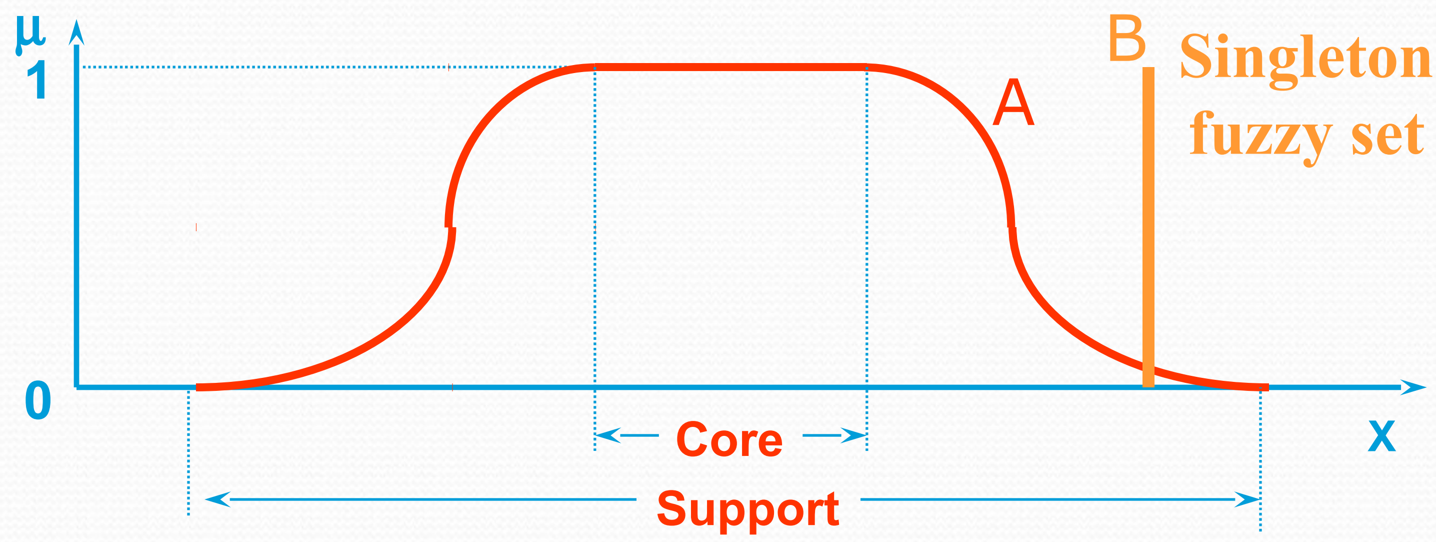
Fuzzy partition

Fuzzy partition formed by the linguistic values "young", "middle aged", and "old":



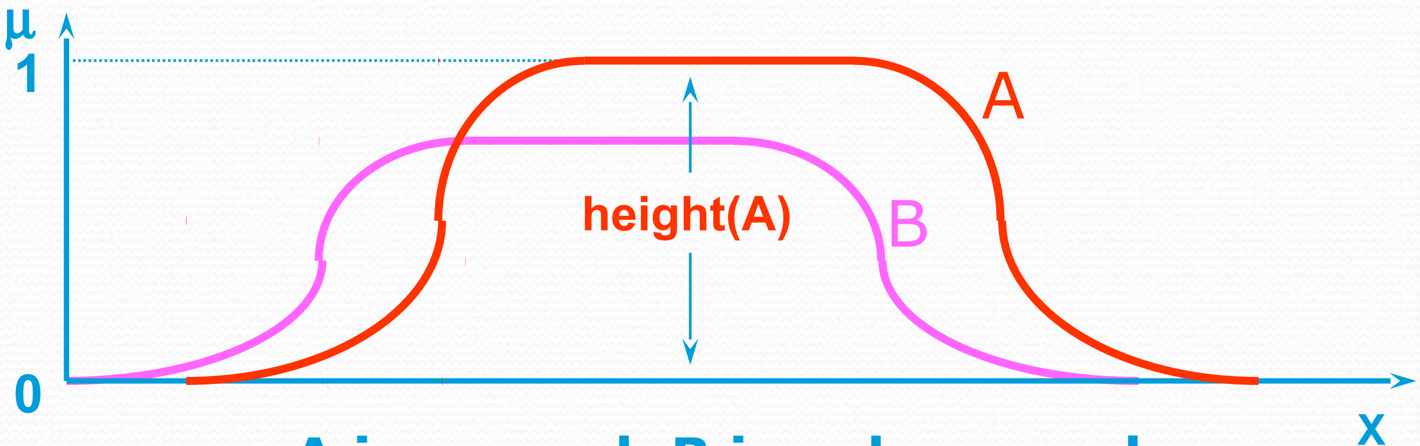
Support, core, singleton

- The *support* of a fuzzy set A in X is the crisp subset of X whose elements have non-zero membership in A: $supp(A) = \{x \in X \mid \mu_A(x)>0\}$
- The *core* of a fuzzy set A in X is the crisp subset of X whose elements have membership 1 in A: $core(A) = \{x \in X \mid \mu_A(x)=1\}$



Normal fuzzy sets

- The *height* of a fuzzy set A is the maximum value of $\mu_A(x)$
- A fuzzy set is called *normal* if its height is 1, otherwise it is called *sub-normal*



A is normal, B is sub-normal

Set theoretic operations /Fuzzy logic connectives

(Specific case)

- Subset:

$$A \subseteq B \Leftrightarrow \mu_A \le \mu_B$$

- Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

- Union:

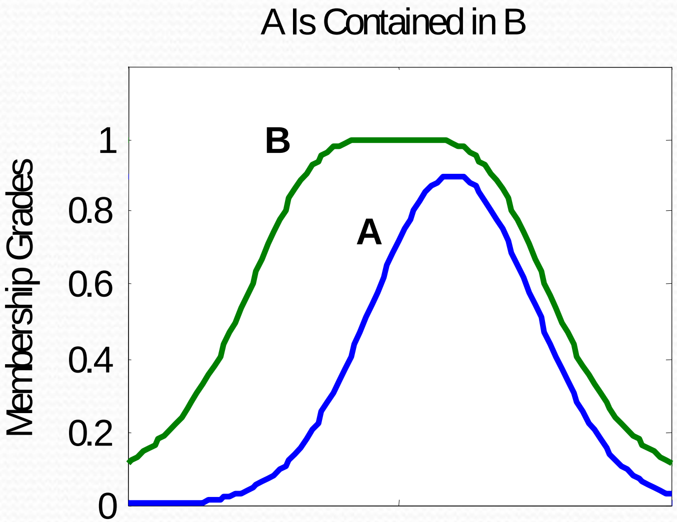
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

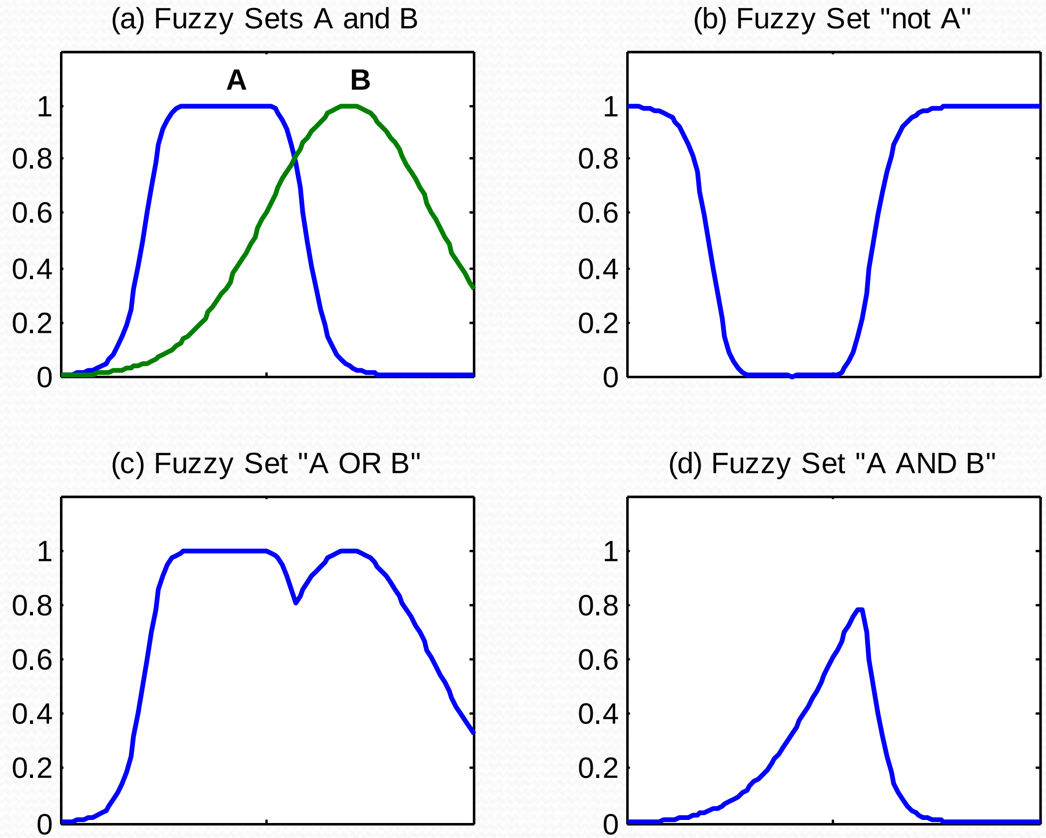
- Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

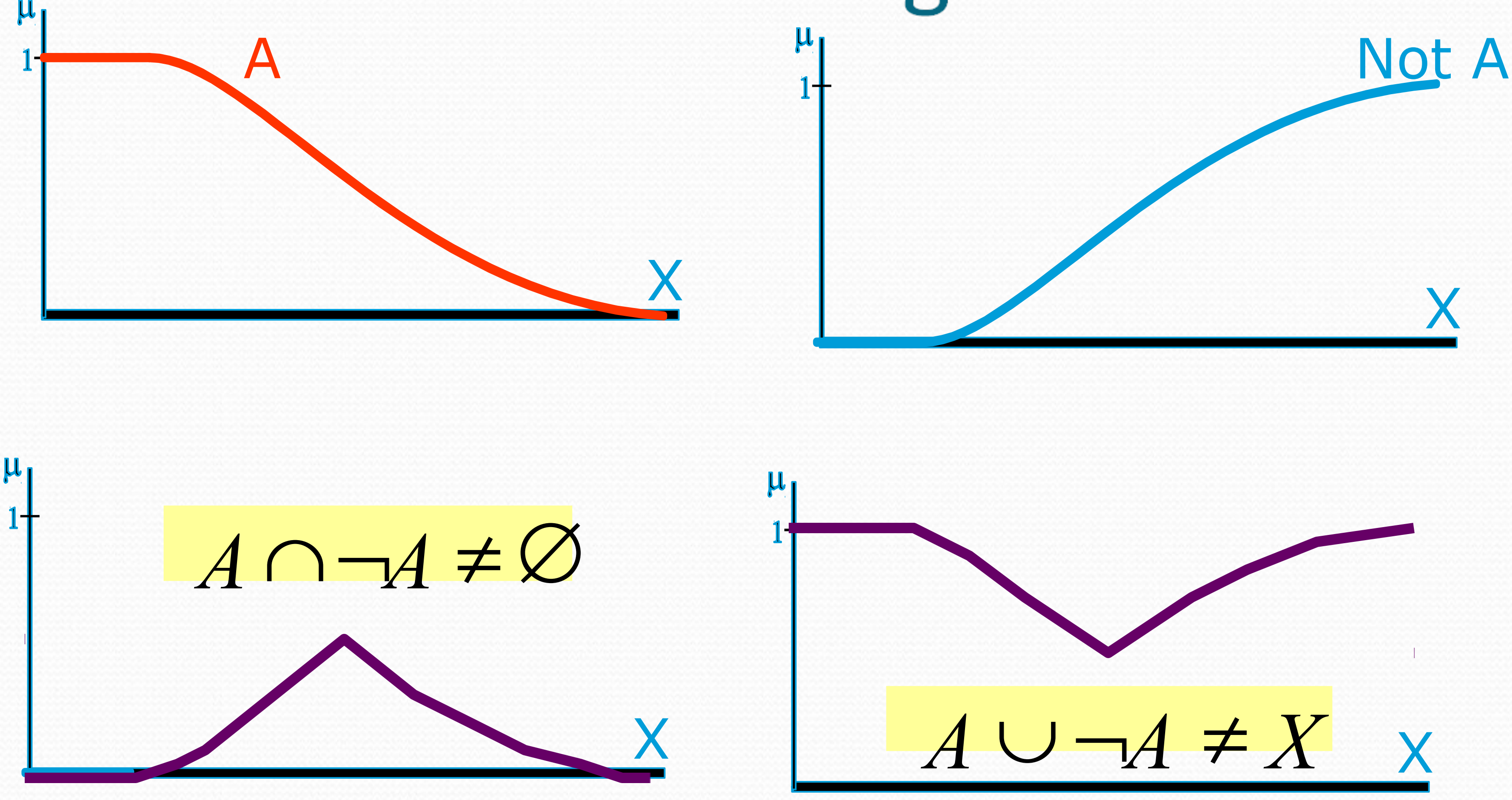
Set theoretic operations

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$$





Combinations with negation



Generalized negation

- General requirements:
  - Boundary: N(0)=1 and N(1) = 0
  - Monotonicity: N(a) > N(b) if a < b
  - Involution: N(N(a)) = a
- Two types of fuzzy complements:
  - Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

  - Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

Sugeno's and Yager's complements

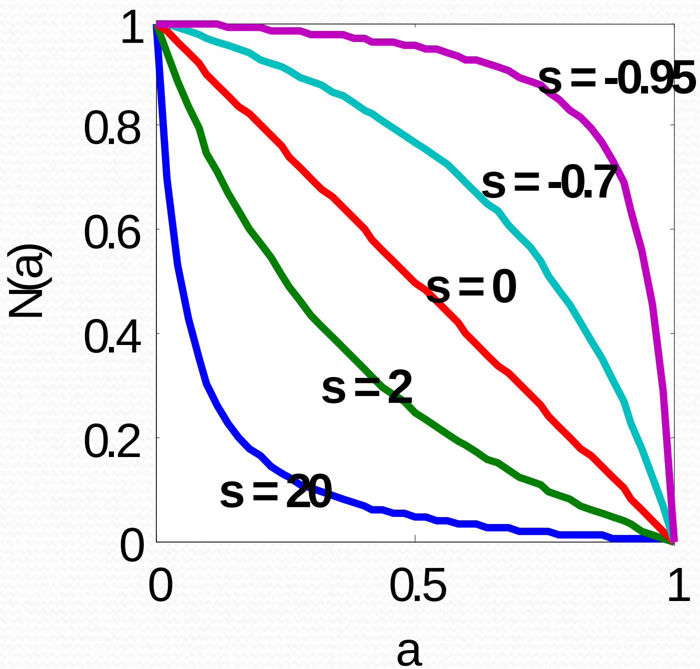
Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

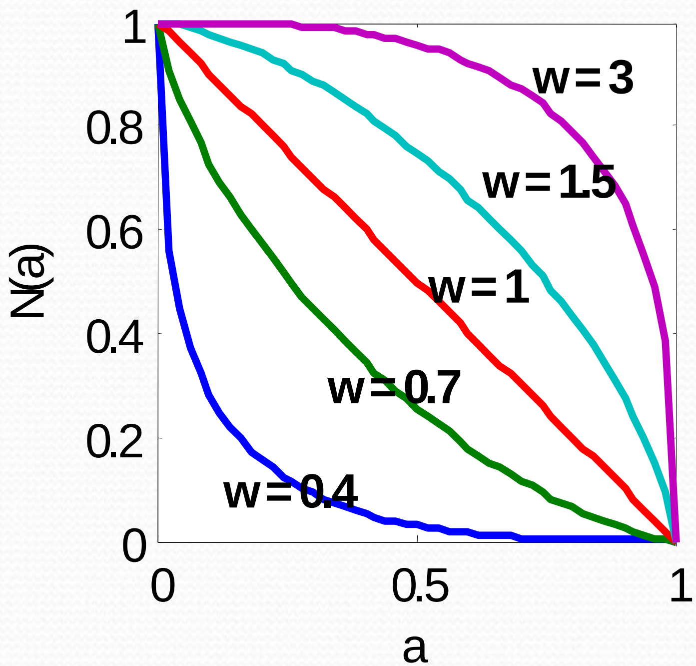
Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

(a) Sugeno's Complements



(b) Yager's Complements



Generalized intersection (Triangular/T-norm, logical and)

- Basic requirements:
  - Boundary: T(0, a) = T(a,0) = 0, T(a, 1) = T(1, a) = a
  - Monotonicity: T(a, b) <= T(c, d) if a <= c and b <= d
  - Commutativity: T(a, b) = T(b, a)
  - Associativity: T(a, T(b, c)) = T(T(a, b), c)

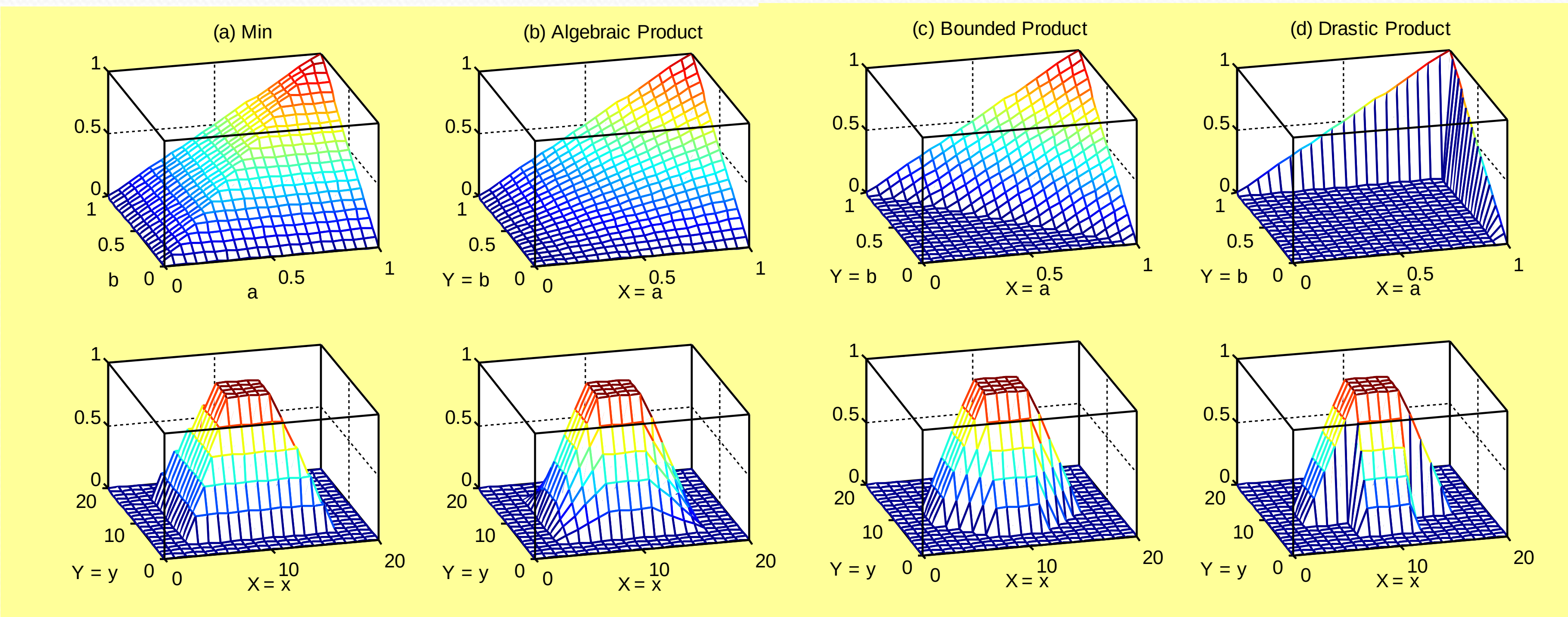
Generalized intersection (Triangular/T-norm)

- Examples:
  - Minimum: $T(a,b) = min(a,b)$
  - Algebraic product: $T(a,b) = a \cdot b$
  - Bounded product: $T(a,b) = max(0,(a+b-1))$
  - Drastic product:

$$T(a,b) = \begin{cases} a & if\ b=1 \\ b & if\ a=1 \\ 0 & otherwise \end{cases}$$

T-norm operator

Minimum: $T_m(a, b)$ $\geq$ **Algebraic product:** $T_a(a, b)$ $\geq$ **Bounded product:** $T_b(a, b)$ $\geq$ **Drastic product:** $T_d(a, b)$

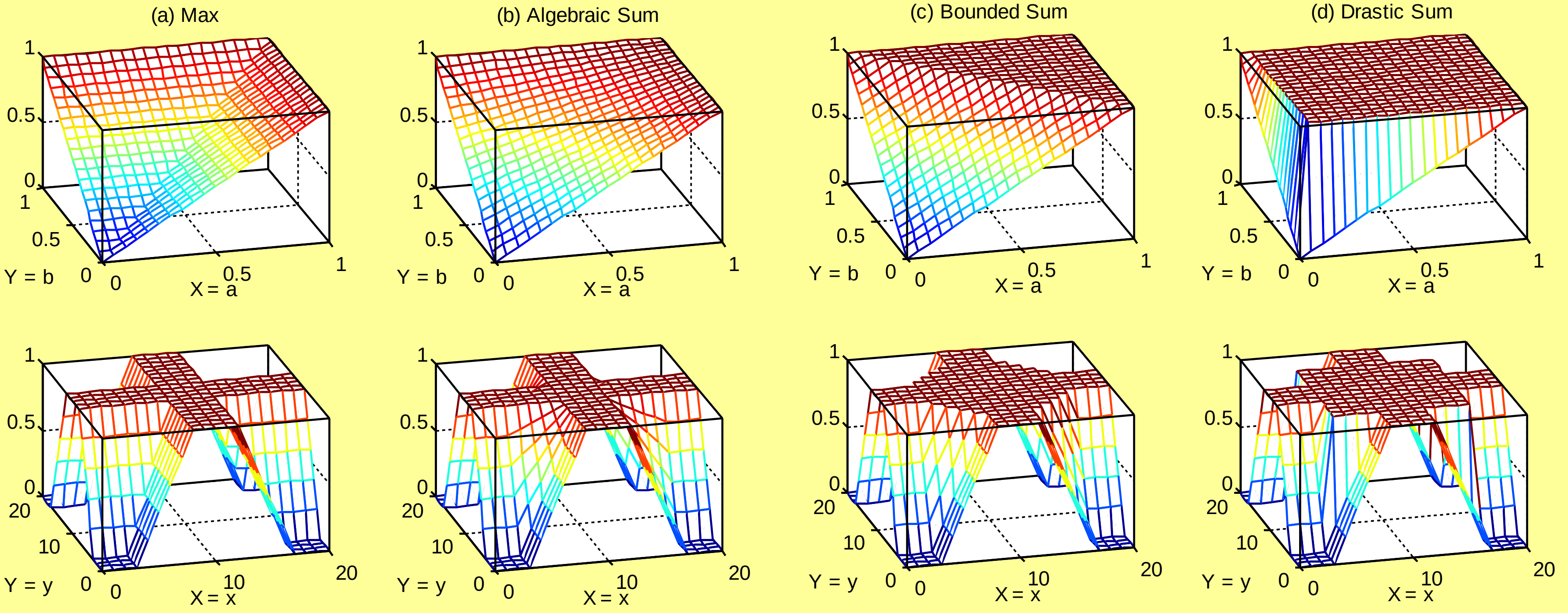


Generalized union (t-conorm)

- Basic requirements:
  - Boundary: S(1, a) = 1, S(a, 0) = S(0, a) = a
  - Monotonicity: S(a, b) < S(c, d) if a < c and b < d
  - Commutativity: S(a, b) = S(b, a)
  - Associativity: S(a, S(b, c)) = S(S(a, b), c)
- Examples:
  - Maximum: $S(a,b) = max(a,b)$
  - Algebraic sum: $S(a,b) = a + b - a \cdot b$
  - Bounded sum: $S(a,b) = min(1,(a+b))$
  - Drastic sum

T-conorm operator

Maximum: $S_m(a, b)$ ≤ **Algebraic sum: $S_a(a, b)$** ≤ **Bounded sum: $S_b(a, b)$** ≤ **Drastic sum: $S_d(a, b)$**



Generalized De Morgan's Law

- T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
  - $T(a, b) = N(S(N(a), N(b)))$
  - $S(a, b) = N(T(N(a), N(b)))$